Recap §1. intro
§2. $\Gamma\backslash G$

today: convergence of Eisenstein series

Let $(G,\Gamma) = (GL_n(\mathbb{R}), GL_n(\mathbb{Z}))$ OR $(SL_n(\mathbb{R}), SL_n(\mathbb{Z}))$

$G = NAK \qquad A = \{\operatorname{diag}(a_1, \dots, a_n) = a \in G : a_j > 0 \ \forall j\}$

$\mathbb{C}^n \ni s = (s_1, \dots, s_n) \rightsquigarrow$ character of A: $a \mapsto a^s = a_1^{s_1} \cdots a_n^{s_n} \in \mathbb{C}^\times = GL_1(\mathbb{C})$

extends to a character of $B = NA$: $ua \mapsto a^s$

trivial on $\Gamma_B := \Gamma \cap B$

Defn We call s dominant if $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2) \geq \dots \geq \mathrm{Re}(s_n)$
strictly dominant if $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) > \dots > \mathrm{Re}(s_n)$

Recall $\delta_B(ua) = \prod_{i<j} a_i/a_j = a_1^{n-1} a_2^{n-3} \cdots a_n^{1-n} = a^{2\rho}$,

$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right)$
(is strictly dominant)

Theorem Let $\mathcal{D} := \{s \in \mathbb{C}^n : s - \rho \text{ is strictly dominant}\}$

The series $E_s(g) := E(s,g) := \sum_{\gamma \in \Gamma_B \backslash \Gamma} a(\gamma g)^{s+\rho}$

converges absolutely, locally uniformly for $(s,g) \in \mathcal{D} \times G$.

$\gamma g = u \cdot a(\gamma g) \cdot k$
$u \in N,\ a(\gamma g) \in A,\ k \in K$

($\Rightarrow s \mapsto E_s(g)$ is holomorphic on $\mathcal{D}$ $\forall g \in G$)

Pf idea: "compare the sum to an integral"

Exercise Assuming the theorem holds for $G = SL_n(\mathbb{R})$, deduce it for $G = GL_n(\mathbb{R})$

Recap on finite-dimensional representation theory of $SL_n(\mathbb{R}) = G$

Defn A <u>weight</u> ω is an equivalence class of elements of $\mathbb{Z}^n$, $\omega_1 \sim \omega_2 \iff \omega_1 - \omega_2 \in \mathbb{Z} \cdot (1, \dots, 1)$.

Thus $\{\text{weights}\} \iff \{\text{polynomial characters of } A \leq SL_n(\mathbb{R})\}$

$\omega \iff [a \mapsto a^{\omega}]$ $\qquad$ a : $a_1 \cdots a_n = 1$, so $a^{\omega_1} = a^{\omega_2}$ if $\omega_1 \sim \omega_2$

Example for $m = 1, \dots, n-1$, $\beta_m := (\underbrace{1, \dots, 1}_{m}, 0, \dots, 0)$ is a nontrivial dominant weight. These are called <u>fundamental weights</u>.

Exercise $\{\text{weights}\} = \bigoplus_{m=1}^{n-1} \mathbb{Z}\beta_m$, $\{\text{dominant weights}\} = \bigoplus_{m=1}^{n-1} \mathbb{Z}_{\geq 0}\beta_m$.

Defn $\omega \geq 0 \overset{\text{DEFN}}{\iff} a^{\omega} \geq 1 \quad \forall a \in A$ s.t. $a_1 \geq \dots \geq a_n$.

$\omega_1 \geq \omega_2 \iff a^{\omega_1} \geq a^{\omega_2}$

$\iff \omega_1 - \omega_2 \geq 0$

Remark ω: dominant $\Rightarrow \omega \geq 0$, but not conversely, $\qquad a^{\omega} = a_1/a_2$

e.g., $\omega = (1, -1, 0) \geq 0$, not dominant.

Defn Let $\sigma : G \to GL(V)$ be a representation (continuous homomorphism) of G on a $\mathbb{C}$-vector space V of finite dimension.
A vector $0 \neq v \in V$ is called a <u>weight vector of weight ω</u> if $\forall a \in A$,

$$\sigma(a) v = a^{\omega} v.$$

Example $V = \mathbb{C}^n$, σ = "identity" $\rightsquigarrow$ "standard representation of G"

$e_1, \dots, e_n$: standard basis vectors are weight vectors of weights $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$.

$$a e_j = a_j e_j$$

Theorem of highest weight $\quad (G = SL_n(\mathbb{R}))$

For each dominant weight ω, $\exists$ finite-dimensional irreducible representation $\nabla_\omega : G \to GL(V_\omega)$ w/ the following properties.

(i) V_ω admits an "integral structure", i.e., $\exists$ $\mathbb{Z}$-module $E_\omega \subseteq V_\omega$ s.t. $V_\omega = E_\omega \otimes_{\mathbb{Z}} \mathbb{C}$ and $\nabla_\omega(SL_n(\mathbb{Z}))E_\omega = E_\omega$.

(ii) V_ω contains a weight vector $e_\omega \in E_\omega$ of weight ω s.t. $\nabla_\omega(ua)e_\omega = a^\omega e_\omega \quad \forall\, ua \in B$.

(i.e., E_ω is N-inv.)

(iii) V_ω admits a basis of weight vectors of weight $\leq \omega$.

(This defines a bijection $\{\text{dominant wts}\} \Leftrightarrow \{\text{f.d. irreps}\}/\sim$.)

Example $\beta_1 = (1,0,\dots,0) \rightsquigarrow (\nabla_{\beta_1}, V_{\beta_1})$: "standard representation"

$$SL_n\mathbb{Z} \curvearrowright \mathbb{Z}^n = E_{\beta_1} \subseteq V_{\beta_1} = \mathbb{C}^n$$

$$E_{\beta_1} \ni e_{\beta_1} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad e_{\beta_1} = e_1$$

e_1 extends to a basis $e_2, \dots, e_n$ of wt vectors, wts $\leq \beta_1$.

Example $\omega = \beta_m = (1,\dots,1,0,\dots,0) \qquad V_\omega = \Lambda^m \mathbb{C}^n$

w/ basis $e_{i_1} \wedge \cdots \wedge e_{i_m} \qquad (1 \leq i_1 < \cdots < i_m \leq n)$

$E_\omega = \mathbb{Z}$-module spanned by this basis

$E_\omega \ni e_\omega = e_1 \wedge \cdots \wedge e_m$ of weight β_m

We equip V_ω with the Euclidean norm $\|\cdot\|$ s.t. this basis is orthonormal. This norm is K-invariant.

Key observation $\quad \forall\, 0 \neq v \in E_\omega, \quad \|v\| \geq 1.$

v: integral linear comb. of the basis elts

Lemma $\forall \gamma \in \Gamma$, $\forall \omega$: dominant weight, we have

$$a(\gamma)^{\omega} \leq 1.$$

Proof We first verify the formula: for $g \in G$, $\omega \in \{\beta_1, \dots, \beta_{n-1}\}$,

$$(*) \qquad a(g)^{\omega} = \| g^{-1} e_\omega \|^{-1}.$$

Indeed, write $g = nak$. $\quad g^{-1} e_\omega = k^{-1} \underbrace{a^{-1} n^{-1} e_\omega}_{= a^{-\omega} e_\omega}$

$\|\cdot\|$: K-inv, so $\quad \|g^{-1} e_\omega\| = \|a^{-\omega} e_\omega\| = a^{-\omega}$. Thus $(*)$ holds.

Now for $\gamma \in \Gamma$, $\quad \| \underbrace{\gamma^{-1} e_\omega}_{\in E_\omega} \| \geq 1$, so $\quad a(\gamma)^{\omega} \leq 1$.

Since any ω lies in $\sum \mathbb{Z}_{\geq 0} \beta_m$, we conclude. $\square$

Example $n = 2$ $\quad SL_2\mathbb{Z} \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Check that

$$a(\gamma) = \begin{pmatrix} 1/\sqrt{c^2+d^2} & \\ & \sqrt{c^2+d^2} \end{pmatrix}.$$

Note $\sqrt{c^2+d^2} \geq 1$

Exercise Let $\mathfrak{S} \subseteq G$ be a Siegel domain.
Let ω : dominant.

Then
$$a(gx)^{\omega} \ll a(g)^{\omega} a(x)^{\omega}. \qquad \forall g \in G, x \in \mathfrak{S}$$

Idea reduce to the case $g \in K$

———— // ———— $g \in$ fixed compact, $x \in A$, $x_1 \geq \dots \geq x_n$

use $(*)$

Cor of Lemma $\forall \gamma \in \Gamma$, we have

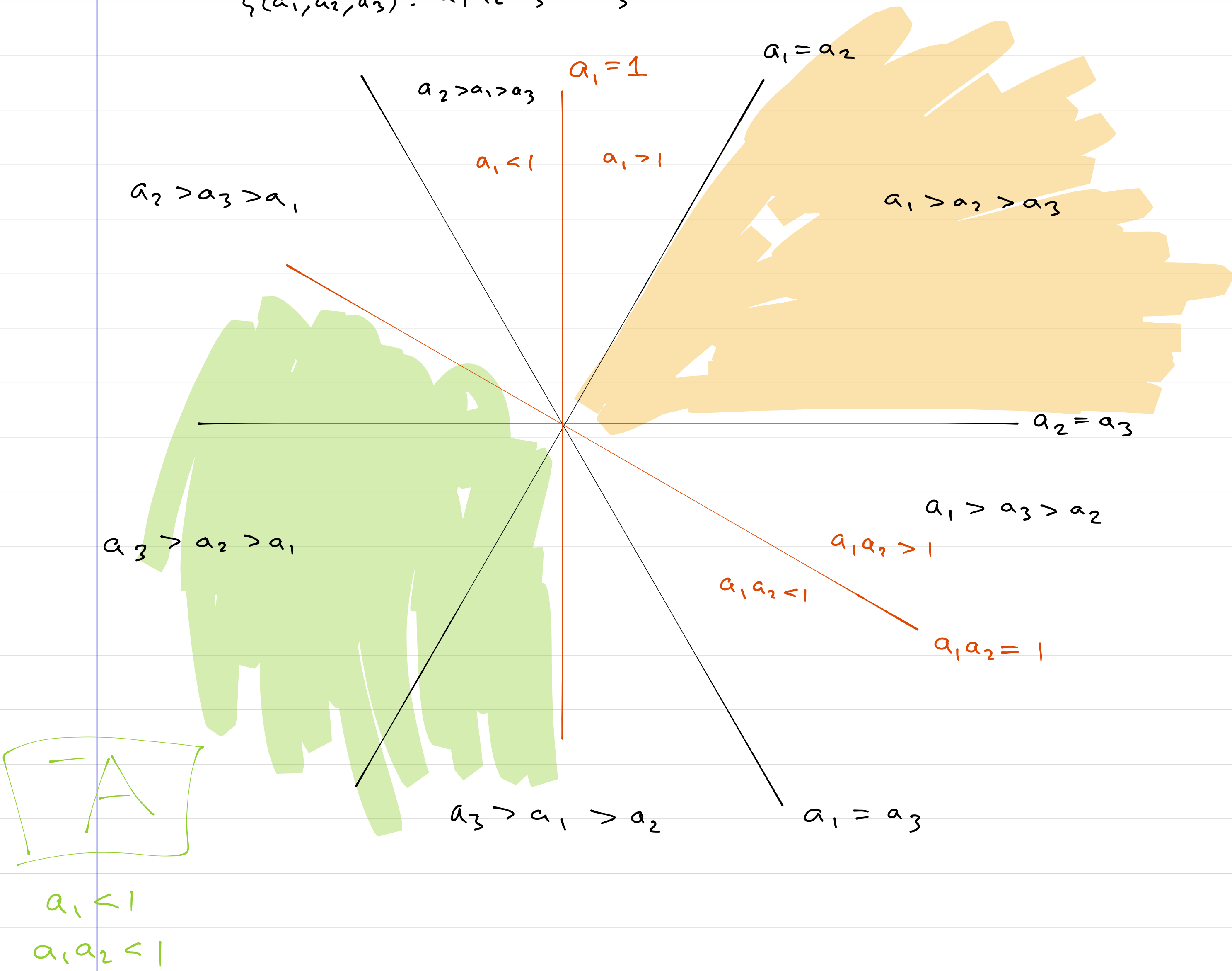
$$a(\gamma) \in {}^{-}A := \{ a \in A : \underbrace{a^{\beta_m}}_{= a_1 \cdots a_m} \leq 1 \ \forall\, m = 1..n-1 \}$$

"negative cone in A"

Map of A for $n=3$

$\shortparallel$

$\{(a_1, a_2, a_3) : a_1 a_2 a_3 = 1\}$



Easy Lemma Let $\Omega \subseteq G$: compact. Let ω: any weight.

Then $\quad a(xg)^{\omega} \underset{\phi}{\asymp} a(x)^{\omega} \quad \forall\, x \in G, \ g \in \Omega$

constants may depend on Ω, ω.

Ex $\Omega = K$

"$\asymp$" $\rightsquigarrow$ "$=$"

Let's now prepare the proof of the Theorem.
Choose a bounded neighborhood Ω of $1 \in G$ s.t.

$$\Omega \cdot \Omega^{-1} \cap \Gamma = \{1\},$$

$$\boxed{E_s(g) = \sum_{\Gamma_B \backslash \Gamma} a(\gamma g)^{s+\rho}}$$

as we may because Γ: discrete in G.

$\Rightarrow$ the translates $\gamma\Omega$ $(\gamma \in \Gamma)$ are disjoint,

$\Rightarrow \underbrace{\Gamma_B \gamma \Omega \ (\gamma \in \Gamma_B \backslash \Gamma)}$ are disjoint.

By "easy lemma", for $g \in$ (fixed compact in G)

$s \in$ (— " — in $\mathcal{D}$)

$\gamma \in \Gamma$, $h \in \Omega$

$$a(\gamma g)^{s+\rho} \asymp a(\gamma)^{s+\rho} \asymp a(\gamma h)^{s+\rho}$$

Also, if $s \in \mathbb{R}^n$, then each of these is > 0.
Thus Theorem will follow if we can show:
$\forall\ s \in \mathbb{R}^n$ s.t. $s - \rho$ strictly dominant,

$$\int_{g \in \Omega} \sum_{\gamma \in \Gamma_B \backslash \Gamma} a(\gamma g)^{s+\rho}\, dg < \infty.$$

<u>Note</u> $\gamma \in \Gamma$, $g \in \Omega \Rightarrow \gamma g \underset{\approx}{\in} N \cdot {}^{-}A \cdot K$

i.e., $\gamma g \in N \cdot a_0 \cdot {}^{-}A \cdot K$

where $a_0 \in A$ depends only on Ω.

Since $a_0^{s+\rho} \asymp 1$, reduce to showing

$$I(s) := \int_{g \in \Gamma_B \backslash N \cdot {}^{-}A \cdot K} a(g)^{s+\rho}\, dg < \infty.$$

fundamental domain where each $|n_{ij}| \leq 1/2$

$\downarrow$

$n \in \Gamma_N \backslash N$: compact

$g = nak \Rightarrow dg = dn \dfrac{da}{\delta(a)} dk$, $\delta(a) = a^{2\rho}$, $k \in K$. — " —

$$\Rightarrow \quad I(s) \ll \int_{a \in {}^{-}A} a^{s+\rho} \frac{da}{a^{2\rho}}$$

$$= \int_{{}^{-}A} a^{s-\rho} \, da.$$

Reduce to:

<u>Lemma</u> if $s \in \mathbb{R}^n$ is strictly dominant, then

$$I := \int_{-A} a^s \, da < \infty. \qquad \left(a = (a_1, \ldots, a_n), \; a_1 \cdots a_n = 1 \right)$$

<u>Proof</u> Use coordinates $t_m := a_1 \cdots a_m \quad (1 \leq m \leq n-1)$
$\in (0,1]$ b/c $a \in {}^{-}A$

Then $$I = \int_0^1 \cdots \int_0^1 t_1^{\nu_1} \cdots t_{n-1}^{\nu_{n-1}} \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}} < \infty$$

<u>Check</u>

$$s_{n-1} = \nu_{n-1} + s_n$$
$$s_{n-2} = \nu_{n-2} + \nu_{n-1} + s_n$$
$$\cdots$$
$$s_1 = \nu_1 + \cdots + \nu_{n-1} + s_n$$

thus s: strictly dominant $\Leftrightarrow$ $\boxed{\nu_j > 0 \quad \forall j}$ $\square$